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Perturbation of relativistic bootstraps

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Abstract. This paper reports the investigation of two problems: (i) the possibility of bootstrapping boundstate scalar and vector mesons from scalar constituents, in an off-shell model, utilizing the direct interaction theory of Bakamjian and Thomas, and then (ii) the effect of applying a linear perturbation procedure to the masses and coupling constants of these solutions. In (i) the Lippmann–Schwinger formalism and an approximate form of crossing symmetry are used to obtain expressions for the bound-state residues, giving solutions to both the scalar and vector cases with properties in excellent agreement with Bethe–Salpeter and N/D dispersion results of other workers. In (ii) we calculate the relative contributions of the 'driving-term' and 'feedback' mechanisms. In the S-wave case we find that the attractive perturbation increases the binding of the bound state, but in the P-wave case, treated with a cut-off, we obtain a 'sign-reversal' of the mass-shift arising from the 'feedback'.

1. Introduction

In the bootstrap approach to π–π systems, one is accustomed to both the N/D formalism of Chew and Mandelstam (1961), Zachariasen (1961), Zachariasen and Zemach (1962), and to the approximation methods of the Bethe–Salpeter equation in, for example, Kaufmann (1968). In this paper, however, we adopt a third approach which commends itself by virtue of the ease and speed with which numerical computations can be carried out.

We use the relativistic direct-interaction formalism of Bakamjian and Thomas (1953), and exploit it via an off-shell equation developed by Fong and Sucher (1964). We consider scalar and vector bootstraps and take as the constituent particles scalar or pseudoscalar mesons. The potentials are provided by σ ($S = 0$, $T = 0$) exchange in the S-wave case and ρ ($S = 1$, $T = 1$) exchange in the P-wave, determined by the leading Feynmann diagrams. (We ignore contributions from other channels.)

We employ the Lippmann–Schwinger formalism to obtain an expression for the residue of the T-matrix pole corresponding to the bound state, and then use an approximate form of crossing-symmetry to obtain an expression for the coupling constant of this bound state. In the P-wave case, we employ a sharp cut-off to avoid the divergence arising from spin 1 exchange. (This method of controlling the divergence is common to many models and we refer to Contogouris *et al.* 1967 for a discussion of the underlying philosophy.)

In both the S- and P-wave case we find two self-consistent solutions, one deeply bound and the other lightly bound. (The P-wave solutions are of course cut-off dependent.)

The existence of S-wave solutions is perhaps surprising since no bound or low-energy resonant σ-particle ($T = 0$, $L = 0$) apparently exists in nature, but work of Kaufmann (1968) with a Bethe–Salpeter equation confirms the heavier solution to within 5%. (Isospin for the S-wave case is ignored here and in Kaufmann.)

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In the P-wave state of π–π, the ρ-meson is of course a resonance in nature, but we have been able to pull it into the bound-state region by decreasing the cut-off down to values around the pion mass; the reason for pulling the ρ out of its physically interesting area is that, in the perturbation procedure which follows, we limit ourselves to considering bound-state structures only.

2. The dynamical equation

Bakamjian and Thomas (1953) have formulated a relativistic theory which, instead of resorting to field operators, is based on a relativistic Schrödinger equation.

The Hamiltonian $\hat{H}$ is a linear operator in a two-particle Hilbert space spanned by plane-wave states $|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle$. For the case of two spinless particles of mass μ_1 and μ_2 the general form of H is

$$\hat{H} = \{(\hat{h})^2 + (\hat{\boldsymbol{P}})^2\}^{1/2}$$

with

$$\hat{\boldsymbol{P}} = \hat{\boldsymbol{p}}_1 + \hat{\boldsymbol{p}}_2$$

$$\hat{h} = E_1(\hat{\boldsymbol{K}}) + E_2(\hat{\boldsymbol{K}}) + \hat{V}$$

and

$$E_i(\boldsymbol{p}) = (\mu_i{}^2 + \boldsymbol{p}^2)^{1/2} \qquad i = 1, 2.$$

The interaction operator $\hat{V}$ introduced in this manner leaves the ten generators of the proper inhomogeneous Lorentz group still satisfying the usual commutation relations of that group. Note that $\hat{\boldsymbol{K}}$ is simply the three-momentum of particle 1 in the instantaneous centre of momentum system. For details of the Bakamjian–Thomas formalism we refer the reader to Bakamjian and Thomas (1953) and Fong and Sucher (1964). Fong and Sucher (1964) show that the B–T Hamiltonian defines a covariant S-matrix.

In this paper we are concerned with bound states and for this case we follow the treatment of Son and Sucher (1967) writing

$$H\boldsymbol{\Psi}_{\mathrm{b},Q} = E_{\mathrm{b}}(Q)\,\boldsymbol{\Psi}_{\mathrm{b},Q} \tag{1}$$

where $\boldsymbol{\Psi}_{\mathrm{b},Q}$ describes a bound state of mass m_{b} and three-momentum $\boldsymbol{Q}$, and where

$$E_{\mathrm{b}}(\boldsymbol{Q}) = (m_{\mathrm{b}}{}^2 + \boldsymbol{Q}^2)^{1/2}.$$

Introducing simultaneous eigenstates $|\boldsymbol{K}, \boldsymbol{P}\rangle$ of $\hat{\boldsymbol{K}}$ and $\hat{\boldsymbol{P}}$ as a basis in $\mathscr{H}$, we define a wave function $\phi_{\mathrm{b}}(k)$ using

$$\langle \boldsymbol{K}, \boldsymbol{P}|\boldsymbol{\Psi}_{\mathrm{b},Q}\rangle = \delta(\boldsymbol{P} - \boldsymbol{Q})\,\phi_{\mathrm{b}}(\boldsymbol{K}).$$

Inserting a complete set of states into equation (1) and putting $|\boldsymbol{Q}| = 0$ gives

$$\{E_1(\boldsymbol{K}) + E_2(\boldsymbol{K}) - m_{\mathrm{b}}\}\phi_{\mathrm{b}}(\boldsymbol{K}) = -\int V(\boldsymbol{K}, \boldsymbol{K}')\,\phi_{\mathrm{b}}(\boldsymbol{K}')\,\mathrm{d}^3\boldsymbol{K}'$$

where

$$V(\boldsymbol{K}, \boldsymbol{K}') = \langle \boldsymbol{K}|\hat{V}|\boldsymbol{K}'\rangle$$

and

$$\int |\phi_{\mathrm{b}}(\boldsymbol{K})|^2\,\mathrm{d}\boldsymbol{K} = 1.$$

With $V(\boldsymbol{K}, \boldsymbol{K}')$ invariant under rotations we have

$$V(\boldsymbol{K}, \boldsymbol{K}') = \sum_l (l + \tfrac{1}{2})\,V_l(k, k')\,P_l(\hat{\boldsymbol{K}} \,.\, \hat{\boldsymbol{K}}')$$

with

$$V_l(k, k') = \int_{-1}^{1} V(k, k'; x) P_l(x) \, dx. \tag{2}$$

For a bound state of angular momentum l we set

$$\phi_b(\boldsymbol{K}) = \phi_l(k) \, Y_l^m(\hat{\boldsymbol{K}})$$

which after a little algebra gives us the equation

$$\{2E_\pi(k) - m_b\}\phi_l(k) = -2\pi \int_0^\infty V_l(k, k') \, \phi_l(k') k'^2 \, dk' \tag{3}$$

where

$$\mu_1 = \mu_2 = \mu_\pi$$

and

$$E_1(\boldsymbol{K}) = E_2(\boldsymbol{K}) = E_\pi = (\mu_\pi^2 + \boldsymbol{K}^2)^{1/2}.$$

Equation (3) is the central equation of our model. We now turn to the problem of determining the potential $V(\boldsymbol{K}, \boldsymbol{K})$.

We introduce the Lippmann–Schwinger formalism for the T-matrix, namely

$$T = V + VGV$$

where

$$G = \frac{1}{(2E_\pi - H + i\mathscr{E})}$$

and H is the B–T Hamiltonian of the last section. T is related to the S-matrix via

$$S^{if} = \delta^{if} - 2\pi i \delta(P_i - P_f) T^{if}.$$

3. Case 1. *S*-wave

For the S-wave bound state we consider the exchange of a neutral scalar particle σ between two non-identical π-mesons. For simplicity, we shall ignore (in this S-wave case) the isospin of real pions. See figure 1.

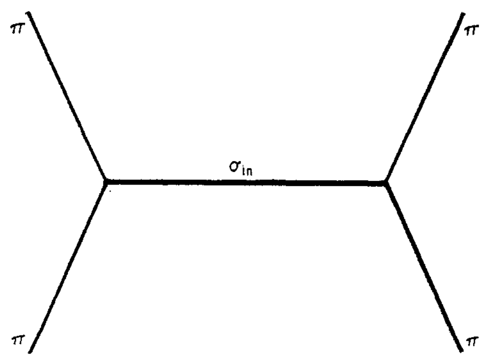

Figure 1. σ-exchange graph.

We use the suffix on σ_{in} to distinguish the exchanged mass from the bound state $m_b = \sigma_{out}$ to be introduced presently.

For the effective Lagrangian at each vertex we use

$$\mathscr{L} = G\phi_\pi\phi_\pi^*\Psi_\sigma.$$

To leading order, the T-matrix is given by

$$T(\mathbf{K}, \mathbf{K}') = \frac{G^2}{4(2\pi)^3}\left(\frac{1}{\mathbf{\Delta}^2-\sigma_{\text{in}}^2}\right)\frac{1}{EE'}$$

where

$$\mathbf{\Delta} = \{(E-E'), \quad (\mathbf{K}-\mathbf{K}')\}$$

and the E, E' factors derive from the usual external line normalization.

Putting $G = 2\mu_\pi g_{\pi\pi\sigma}$ and using the Born approximation to the scattering amplitude, i.e.

$$V = T^{\text{Born}}$$

we obtain

$$V(\mathbf{K}, \mathbf{K}') = \frac{-g_{\pi\pi\sigma}{}^2}{(2\pi)^3 EE'}\left(\frac{1}{\mathbf{\Delta}^2+\sigma_{\text{in}}^2}\right).$$

Using equation (2) to project out the $l = 0$ partial wave gives

$$V_0(k, k'; \sigma_{\text{in}}) = \frac{-g_{\pi\pi\sigma}{}^2}{2(2\pi)^3 kk'}\left[\ln\left\{\frac{(k+k')^2+\sigma_{\text{in}}^2}{(k-k')^2+\sigma_{\text{in}}^2}\right\}\right]\frac{1}{EE'}.$$

For the bound state we put $m_{\text{b}} = \sigma_{\text{out}}$ and enforcing the bootstrap condition gives

$$\sigma_{\text{in}} = \sigma_{\text{out}} = \sigma.$$

Equation (3) is now

$$(2E_\pi-\sigma)\,\phi_0(k) = -2\pi\int_0^\infty V_0(k, k', \sigma)\,\phi_0(k')k'^2\,\mathrm{d}k'. \tag{4}$$

We solve this equation by numerical methods on the computer (see Appendix 1 for details), putting $\lambda_{\text{in}} = g^2/4\pi$, setting $\mu_\pi = 1$ once and for all and writing equation (4) as

$$\lambda_{\text{in}}{}^{-1}\phi_0 = \hat{\mathscr{R}}\phi_0$$

where $\lambda_{\text{in}}{}^{-1}$ is the eigenvalue of the operator $\hat{\mathscr{R}}$. The lowest value of λ_{in} is sought for with σ in the range $0 < \sigma < 2$. This provides us with a plot of λ_{in} against σ.

These bound states now manifest themselves as poles in the T-matrix such that for

$$\hat{T} = \hat{V}+\hat{V}\hat{G}\hat{V}$$

$$\hat{G} = \frac{1}{2E_\pi-H+\mathrm{i}\mathscr{E}}$$

and $H\mathbf{\Psi}_{\text{b}} = \sigma\mathbf{\Psi}_{\text{b}}$, i.e. $H|B\rangle = \sigma|B\rangle$ we can write near the pole

$$\langle \mathbf{K}'|\hat{T}|\mathbf{K}\rangle \simeq \frac{\langle \mathbf{K}'|\hat{V}|B\rangle\langle B|\hat{V}|\mathbf{K}\rangle}{(2E_\pi-\sigma)}.$$

The residue of the pole gives

$$\text{Res}\; T_0(k, \sigma, \lambda_{\text{in}}) = |\langle \mathbf{K}|\hat{V}|B\rangle|^2_{2E_\pi=\sigma}. \tag{5}$$

Knowing λ_{in} and ϕ_0 for each σ gives Res T_0 for each σ.

Note

$$\langle \boldsymbol{K}|\hat{V}|B\rangle = \int V(\boldsymbol{K}, \boldsymbol{K}')\, \phi_b(\boldsymbol{K}')\, d^3\boldsymbol{K}'$$

where $|\boldsymbol{K}|$ is fixed by $\sigma^2 = 4(k^2+1)$.

We constrain the continuous set of solutions for σ by appealing to an approximate form of crossing symmetry.

If M is the invariant scattering amplitude such that in the region of the pole

$$M(s, t) \sim \left.\frac{G^2}{(t-\sigma^2)}\right\} \quad \begin{aligned} s &= 4(k^2+1) \\ t &= -2k^2(1-\cos\theta) \end{aligned}$$

then

$$M = 32\pi^3 E_\pi{}^2 T.$$

Under crossing $s \leftrightarrow t$ and putting $\lambda_{\text{out}} = g^2/4\pi = G^2/16\pi$ gives

$$\text{Res}\, M_0(s, t) = 8\pi^3\sigma^2\, \text{Res}\, T_0.$$

With equation (5) this provides us finally with

$$\lambda_{\text{out}} = \pi^2\sigma^3 |\langle \boldsymbol{K}|\hat{V}|B\rangle|^2_{2E_\pi=\sigma}. \tag{6}$$

Plotting λ_{out} and λ_{in} against σ gives the curves shown in figure 2.

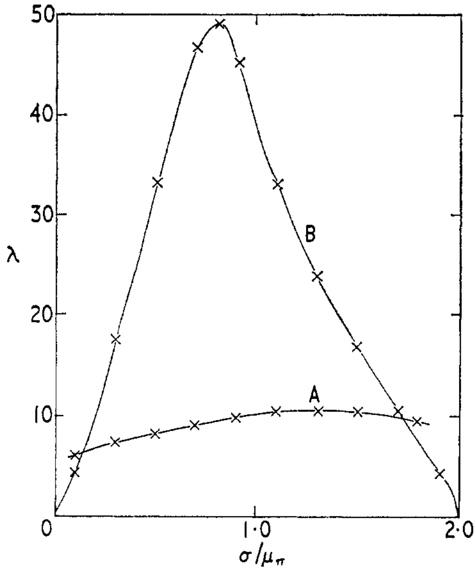

Figure 2. λ_{in} (curve A) and λ_{out} (curve B) against σ/μ_π.

For the bootstrap we must have

$$\lambda_{\text{in}} = \lambda_{\text{out}} = \lambda$$

i.e. the self-consistent scalar bootstrap solutions are the *intersections* of the curves λ_{in} and λ_{out}.

It will be noticed that $\lambda_{\text{out}} \to 0$ as $\sigma_{\text{out}} \to 2$, in agreement with the non-relativistic zero-range approximation for S-waves. We have checked numerically that the standard formula is satisfied by our solutions.

For the lightly bound solution we have

$$\text{Solution 2}\quad \begin{cases}\lambda_2 = 9{\cdot}88\,312 \pm 0{\cdot}00\,005\\ \sigma_2 = 1{\cdot}718\,070 \pm 0{\cdot}000\,005.\end{cases}$$

Comparing with other models we quote the result of Kaufmann (1968) using a Schwinger variational method and the Bethe–Salpeter equation. Kaufmann obtains

$$\lambda = 9{\cdot}42$$
$$\sigma = 1{\cdot}76.$$

This agrees with the above solution to within 5%. What we would expect, *a priori*, is close but not exact agreement, since the physics is the same but the mathematics is somewhat different.

For the strongly bound solution we find

$$\text{Solution 1}\quad \begin{cases}\lambda_1 = 6{\cdot}235 \pm 0{\cdot}005\\ \sigma_1 = 0{\cdot}136 \pm 0{\cdot}005.\end{cases}$$

Although we have ignored isospin in this S-wave case we note some numerical coincidences in our solutions. In Solution 2 the values of λ_2 and σ_2 differ from π^2 and $(\mathrm{e}-1)$ by one part in 1000 and two parts in 17 000 respectively, where e is the base of the natural logarithm, and if in Solution 1 we put $\mu_\pi = 139{\cdot}6$ MeV we obtain $\sigma_1 = 19{\cdot}00 \pm 0{\cdot}68$ MeV. This is to be compared with $\sigma = 19{\cdot}57$ MeV quoted by Aron (1968) in a scheme which generates the $I = 0$, $Y = 0$ particles as simply $m = n\sigma$ where n is an integer. We refer to his paper for details.

4. Case 2. *P*-wave

Experiment confirms a strong resonant behaviour in π–π scattering around 760 MeV. This ρ-meson is of course a resonance but we hope to pull it into the bound-state region (in preparation for the perturbation calculation) by suitable adjustment of the cut-off parameter. That we anticipate success in this venture is based on the work of other workers, e.g. Contogouris *et al.* (1967), who use an N/D model with a left-hand discontinuity defined by the exchange of a vector meson. Using a sharp cut-off to suppress the distant part, they obtain a set of solutions from just above threshold to $m_\rho{}^2 \simeq 30\mu_\pi{}^2$ where the cut-off Λ is given by $11\mu_\pi{}^2 \leqslant \Lambda \leqslant 50\mu_\pi{}^2$.

We therefore expect to obtain solutions below threshold for values of $\Lambda \leqslant 11\mu_\pi{}^2$.

The P-wave analysis follows similar lines to the S-wave case and so we shall only dwell on points of divergence from that case.

5. *P*-wave analysis and results

We consider two pseudoscalar $T = 1$ pions with an attractive $T = 1$, $J = 1$ force (the ρ-meson) generating, we hope, the same $T = 1$, $J = 1$ ρ-meson as a bound state.

The effective interaction at each vertex is

$$\mathscr{L} = \mathrm{i}g_{\pi\pi\rho}\mathscr{E}_{ijk}\rho\pi^\mu{}_j\partial_\mu\pi_k$$

where i, j, k are isotopic indices. The polarization vector $\boldsymbol{\epsilon}_\mu$ of the ρ satisfies

$$\sum_{\text{pol}} \boldsymbol{\epsilon}^\mu\boldsymbol{\epsilon}_\nu = g^\mu_\nu - \frac{\boldsymbol{\Delta}^\mu\boldsymbol{\Delta}_\nu}{\rho^2}$$

with

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}.$$

To leading order in the scattering and taking as before, the Born approximation to the T-matrix gives

$$V(\boldsymbol{K}, \boldsymbol{K}') = -\frac{\lambda^{II'}}{8\pi^2 EE'}\left\{\frac{(E+E')^2+(\boldsymbol{K}+\boldsymbol{K}')^2}{(\boldsymbol{K}-\boldsymbol{K}')^2+\rho_{\text{in}}^2}\right\}.$$

The coupling constants satisfy SU(2) so that

$$\lambda^{II'} = \lambda_{\text{in}} \begin{pmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix} \{1+(-1)^{I+I'}\}$$

where I is the isospin of the external particles and I' is the isospin of the exchanged particle.

Projecting out the $l = 1$ partial wave gives the P-wave potential:

$$V_1(k, k') = \frac{\lambda_{\text{in}}\{(E+E')^2+2(k^2+k'^2)+\rho_{\text{in}}^2\}}{8\pi^2 EE'(2kk')}\left\{2 - \alpha \ln\left(\frac{\alpha+1}{\alpha-1}\right)\right\}$$

where

$$\alpha = \frac{k^2+k'^2+\rho_{\text{in}}^2}{2kk'}.$$

Putting $m_b = \rho_{\text{out}}$ in equation (3) gives

$$(2E_\pi - \rho_{\text{out}})\,\phi_1(k) = -2\pi \int_0^\infty V_1(k, k')\,\phi_1(k')k'^2\,\mathrm{d}k'.$$

Inspection shows that the kernel of this equation, with $V_1(k, k')$ inserted, is not Fredholm, so we write

$$V_1(k, k') \to \Theta(\Lambda - k)\,V_1(k, k')\,\Theta(\Lambda - k')$$

$$\Theta\left\{\Lambda - \begin{pmatrix} k \\ k' \end{pmatrix}\right\}\begin{cases} = 1 \text{ for } \begin{pmatrix} k \\ k' \end{pmatrix} < \Lambda \\ = 0 \text{ for } \begin{pmatrix} k \\ k' \end{pmatrix} > \Lambda. \end{cases}$$

Here Λ has the dimensions of mass.

As in the S-wave case we calculate the lowest value of λ_{in} for each $\rho_{\text{in}} = \rho_{\text{out}} = \rho$. The P-wave residue is given by the analogue of equation (5).

The M-matrix is given as

$$M(t, s, u) \sim \left(\frac{s-u}{t-\rho^2}\right).$$

Under crossing $s \leftrightarrow t$

$$M(s, t, u) \sim \left(\frac{t-u}{s-\rho^2}\right)$$

and projecting out the $l = 1$ wave, gives for the residue of the T-matrix

$$\text{Res } T_1(s, t) = -\frac{g^2 q^2}{12\pi^3 \rho^3}$$

where $q = (1-\rho^2/4)^{1/2}$. We finally obtain for $\lambda_{\text{out}}(= g^2/4\pi)$ the expression

$$\lambda_{\text{out}} = -\frac{3\pi^3\rho^3}{q^2}\left\{\int_0^{\Lambda} V_1(\lambda_{\text{in}}, k = \text{i}q, k')k'^2 \phi_1(k')\, \text{d}k'\right\}^2.$$

Plotting λ_{out} and λ_{in} against ρ, we look for bootstrap solutions $\lambda_{\text{in}} = \lambda_{\text{out}} = \lambda$ in the cut-off range $1 \leqslant \Lambda \leqslant 15$.

Within this range we find a deeply bound solution analogous to the S-wave situation, but the second, more lightly bound solution, does not appear at threshold ($\rho = 2{\cdot}0$) until Λ is reduced to the value of 3·30. This agrees very well with the first resonant solution of Contogouris *et al.* (1967) with their values of $\rho = 2{\cdot}04$, $\Lambda = 3{\cdot}30$. With $\Lambda = 1$ we obtain figure 3.

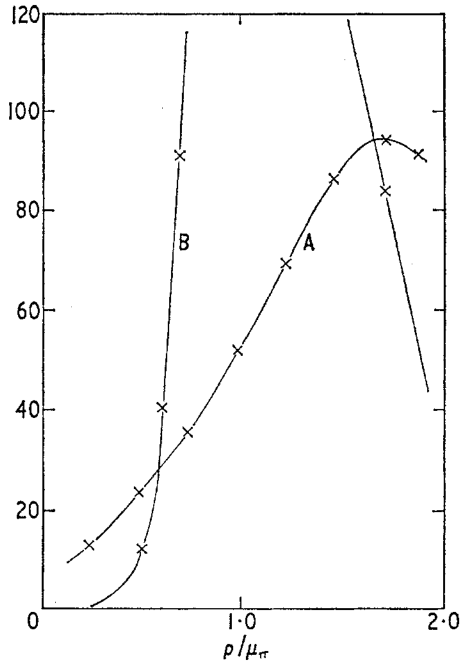

Figure 3. λ_{in} (curve A) and λ_{out} (curve B) against ρ/μ_π.

The agreement with the above workers provides a very satisfying result, showing that their N/D dispersion–theoretical model and our off-shell model are evidently dealing, with similar adequacy, with similar physics.

6. Introduction to the perturbation problem

With these scalar and vector bootstrap bound states we now proceed to investigate the effect of applying a small perturbation to the masses and coupling constants. Our interest in doing this centres on discerning two separate effects. The first is the direct effect of the applied external perturbation on the eigenvalue, i.e. the mass: we shall call this the 'driving-term'. The second effect concerns the response of the strong interaction itself to the presence of the perturbation; this effect manifests itself in a further contribution to the mass shift, and we shall refer to this as the 'feedback effect'. We shall be interested in comparing the differences (should there be any) in the scalar and vector cases.

For the mass shift of the ith particle we can write under perturbation

$$\delta M_i = \sum_j A_{i\ j}^{MM}\,\delta M_j + \sum_j A_{i\ j}^{M\lambda}\,\delta\lambda_j + D_i^M \tag{7}$$

where the A_{ij} are 'feedback' coefficients registering the response of the strong interaction to the perturbation, and the D_i^M the driving terms describing the direct effect of this perturbation.

Since we have considered bootstrapped σ and ρ particles, the sums in equation (7) contain only one term, namely

$$\delta M_i = \frac{D_i^M + A_{i\ i}^{M\lambda}\,\delta\lambda_i}{(1-A_{i\ i}^{MM})}. \tag{8}$$

The sign of δM_i obviously depends upon the signs of D_i^M and the feedback coefficients $A_{i\ i}^{M\lambda}$ and $A_{i\ i}^{MM}$. The controversy over the neutron–proton mass difference, for example, has centred partly on whether the positive sign for $M_N - M_P$ is due to the driving terms or the feedback effects. For references see Dashen and Frautshi (1964) and Barton (1966).

In equation (8) what sign do we expect $A_{i\ i}^{MM}$ to have?

This term measures the effect on the bound state resulting from a change in the mass of the exchanged particle responsible for the strong interaction.
i.e.

$$A^{MM} = \frac{\mathrm{d}m_\mathrm{b}}{\mathrm{d}m_\mathrm{e}} \qquad \begin{cases} \mathrm{b} = \text{bound mass} \\ \mathrm{e} = \text{exchanged.} \end{cases}$$

For the S-wave potential, transcribed into coordinate space in an obvious non-relativistic limit, we have

$$V(r) \simeq -\frac{g^2 \exp(-\sigma_{\mathrm{in}} r)}{4\pi r}.$$

For

$$m_\mathrm{e} = \sigma_{\mathrm{in}} \to \sigma_{\mathrm{in}} + \delta\sigma_{\mathrm{in}}$$

we obtain

$$\delta V(r) \simeq +\frac{g^2}{4\pi}\exp(-\sigma_{\mathrm{in}} r)\delta\sigma_{\mathrm{in}}.$$

This is everywhere positive and therefore repulsive for all r. This decreases the binding so that $m_\mathrm{b}(=\sigma_{\mathrm{out}})$ would be expected to increase, and we would expect

$$A^{MM} = \frac{\mathrm{d}\sigma_{\mathrm{out}}}{\mathrm{d}\sigma_{\mathrm{in}}} > 0.$$

For the P-wave potential we have

$$V(r) \sim \frac{g^2}{4E^2}\left\{\delta(r) - \frac{(8k^2+4\mu_\pi{}^2+\rho_{\text{in}}{}^2)}{4\pi}\frac{\exp(-\rho_{\text{in}}r)}{r}\right\}.$$

With $m_e = \rho_{\text{in}} \to \rho_{\text{in}}+\delta\rho_{\text{in}}$, and neglecting $\delta(r)$ (since $\phi_1(r) = 0$ at $r = 0$), we find

$$\delta V(r) \simeq \frac{g^2}{16\pi E^2}\left(8k^2 + 4\mu_\pi{}^2+\rho_{\text{in}}{}^2 - \frac{2\rho_{\text{in}}}{r}\right)\exp(-\rho_{\text{in}}r)\delta\rho_{\text{in}}.$$

For large r, $\delta V(r)$ is positive and therefore repulsive but, at small r, $\delta V(r)$ is negative, causing attraction. The effect on m_b ($= \rho_{\text{out}}$) now depends on how strongly the state is bound. (See also Kayser 1967.)

If ρ is strongly bound, $\phi_1(r)$ is effectively confined near the origin and there it will feel directly the attractive part of δV. For ρ lightly bound, however, $\phi_1(r)$ will have a long tail and will respond also to the repulsive part of δV.

The sign of $A^{MM} = \mathrm{d}\rho_{\text{out}}/\mathrm{d}\rho_{\text{in}}$ must therefore await explicit calculation.

The results we shall obtain are

$$S\text{-wave } A^{MM}_{\sigma\,\sigma} = +0{\cdot}60$$

$$P\text{-wave } A^{MM}_{\rho\,\rho} = +1{\cdot}16.$$

We shall find that the P-wave is almost cut-off-independent for our bound solutions.

Since $A^{MM}_{\rho\rho} > 1$ we have $(1-A^{MM}_{\rho\rho}) < 0$, providing a mechanism for changing the sign of the mass shift as dictated by the driving term.

7. Perturbation formalism

We now turn to the perturbation procedure within the Bakamjian–Thomas framework.

We recall from equation (1) the Bakamjian–Thomas operator

$$\hat{h} = 2E(\hat{\boldsymbol{K}})+\hat{V}$$

where

$$E(\hat{\boldsymbol{K}}) = (\hat{\boldsymbol{K}}^2+\mu_\pi{}^2)^{1/2}.$$

Under the application of a small perturbation δV^c we obtain

$$\hat{h} \to 2\hat{E}+\hat{V}+\delta\hat{V}+\delta V^c$$

and

$$N|\boldsymbol{\Psi}\rangle \to N'(|\boldsymbol{\Psi}\rangle+|\delta\boldsymbol{\Psi}\rangle)$$

where N' is the new normalization constant.

Remembering that $\hat{h}|\boldsymbol{\Psi}\rangle = m_b|\boldsymbol{\Psi}\rangle$ and $\langle\boldsymbol{K}|\boldsymbol{\Psi}\rangle = \phi_b(\boldsymbol{K}) = \phi_l Y_l^m$ we obtain an inhomogeneous integral equation for $\delta\phi_l$ whose solution is our aim.

$$\hat{U}_l\,\delta\phi_l = \delta m_b - \delta\lambda_{\text{in}}\,\gamma_1^l - \delta m_e\,\gamma_2^l - \gamma_3^l. \tag{9}$$

In equation (9), $\hat{U}_l$ is the homogeneous operator satisfying $\hat{U}_l\phi_l = 0$ and the γ_n^l (functions of the form $\langle\boldsymbol{K}|\delta\hat{V}|\boldsymbol{\Psi}\rangle$) are obtained by using

$$\delta V_l = \frac{\partial V_l}{\partial m_e}\delta m_e + \frac{\delta V_l}{\partial\lambda_{\text{in}}}\delta\lambda_{\text{in}}.$$

In order to solve equation (9) we need expressions for δm and $\delta\lambda$—the shifts in the masses and coupling constants. For the first we use the unitarity relation $\langle\phi|\phi\rangle = 1$ and the bootstrap relation $\delta m_b = \delta m_e = \delta m$ giving

$$\delta m = \frac{\langle\delta V_l^c\rangle + \langle\partial V_l/\partial\lambda_{in}\rangle\delta\lambda_{in}}{(1-\langle\partial V/\partial m_e\rangle)}. \tag{10}$$

Comparing with equation (8) we identify the driving term $D_l^M = \langle\delta V_l^c\rangle$ and the feedback coefficients

$$A_{l\ l}^{M\lambda} = \left\langle\frac{\partial V_l}{\partial\lambda_{in}}\right\rangle$$

$$A_{l\ l}^{MM} = \left\langle\frac{\partial V_l}{\partial m_e}\right\rangle.$$

The second constraint we have is the relationship between the input and output residues of the bound-state poles, namely equations (5) and (6). For the purposes of differentiation, λ_{in} and λ_{out} are kept distinct under the perturbation and then finally identified under the bootstrap condition $\delta\lambda_{in} = \delta\lambda_{out} = \delta\lambda$. This calculation of $\delta\lambda$ involves a lot of tedious algebra and for details refer to Wallace (1968); the form of $\delta\lambda$ finally obtained is

$$\delta\lambda = A_l + B_l\,\delta\phi_l. \tag{11}$$

With these expressions for δm and $\delta\lambda$ we can insert them into equation (9) giving us the equation

$$(\hat{U}_l + \hat{W}_l)\,\delta\phi_l = Q_l \tag{12}$$

where

$$\hat{W}_l\,\phi_l \equiv 0.$$

Since $\hat{U}_l\phi_l = 0$, $\hat{U}_l + \hat{W}_l$ is a singular Hermitian operator and as such will only have solutions if $\langle\phi_l|Q_l\rangle = 0$.

For the proof of this assertion and the remaining details about the method of solving the inhomogeneous equation for $\delta\phi_l$, the reader is referred to Appendix 2.

$\delta\phi_l$ once obtained is inserted into equations (10) and (11) to give δm and $\delta\lambda$.

8. Results

For the perturbing potential we take the relativistic scalar exchange

$$\delta V^c(\boldsymbol{K}, \boldsymbol{K}') = -\frac{\alpha}{2\pi^2 E_\pi E_\pi'}\left\{\frac{1}{(\boldsymbol{K}-\boldsymbol{K}')^2+\nu^2}\right\} \tag{13}$$

where α measures the strength of the perturbation (we are performing a calculation linear in α), and where we explore the consequences of a range of exchanged 'photon masses' ν.

The S- and P-wave projections are simply

$$\delta V_0^c = -\frac{\alpha}{4\pi^2 kk'EE'}\ln\left\{\frac{(k+k')^2+\nu^2}{(k-k')^2+\nu^2}\right\}$$

$$\delta V_1^c = \frac{\alpha}{4\pi^2 kk'EE'}\left[2-\left(\frac{k^2+k'^2+\nu^2}{2kk'}\right)\ln\left\{\frac{(k+k')^2+\nu^2}{(k-k')^2+\nu^2}\right\}\right].$$

The negative sign in equation (13) implies attraction.

Invoking the formalism of the previous section and, in particular, equations (10), (11) and (12), we calculate δm and $\delta\lambda$ for a range of 'photon masses'

$$0 < \nu \leqslant 1{\cdot}6\mu_\pi.$$

For the S-wave the results are displayed in figures 4 and 5.

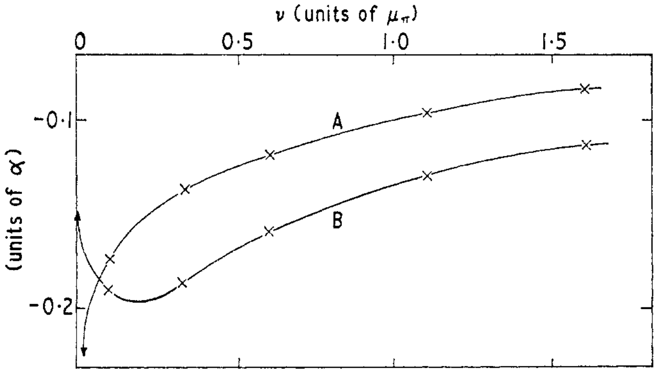

Figure 4. $\delta\sigma/\sigma$ (curve A) and $\delta\lambda/\lambda$ (curve B) against ν.

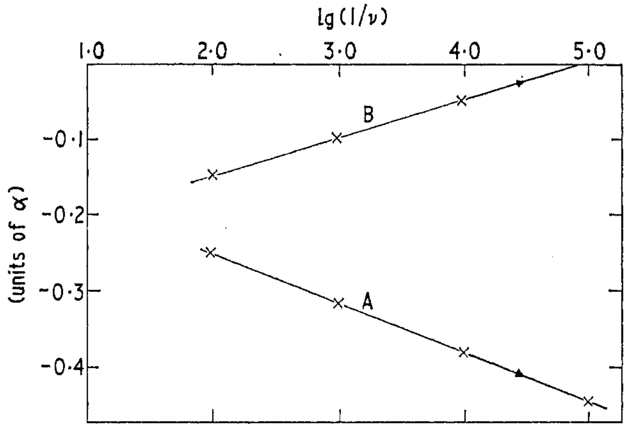

Figure 5. $\delta\sigma/\sigma$ (curve A) and $\delta\lambda/\lambda$ (curve B) against lg $(1/\nu)$.

For the 'heavy-photon' region, i.e. figure 4, the mass shift $\delta\sigma/\sigma$ is dominated by the negative (attractive) driving term resulting from direct 'photon exchange'.

From equation (10) we can write

$$\delta\sigma \sim \frac{\langle\delta V_0{}^{c}\rangle}{(1-\langle\partial V_0/\partial m_e\rangle)}.$$

The 'feedback' term $\langle\partial V_0/\partial m_e\rangle$ is calculated to be

$$\langle\partial V_0/\partial m_e\rangle = 0{\cdot}6$$

so that $\delta\sigma$ takes the same sign as $\langle\delta V_0{}^{c}\rangle$ (i.e. negative).

For the mass of the 'photon' going to zero, i.e. figure 5, we obtain logarithmic behaviour, in an 'infrared' divergence, for both $\delta\sigma/\sigma$ and $\delta\lambda/\lambda$. This infrared divergence enters the system via the coupling constant shift and the linear nature of the perturbation procedure. What is surprising, however, is that the mass shift should also pick up this divergence. The explanation lies in the self-bootstrapping nature of our bound-state model and is reflected in equation (10), connecting $\delta\sigma$ with $\delta\lambda$.

For the P-wave we have the results of figures 6 and 7.

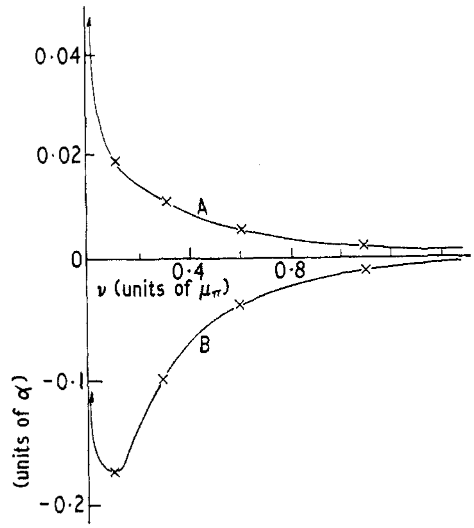

Figure 6. $\delta\rho/\rho$ (curve A) and $\delta\lambda/\lambda$ (curve B) against ν.

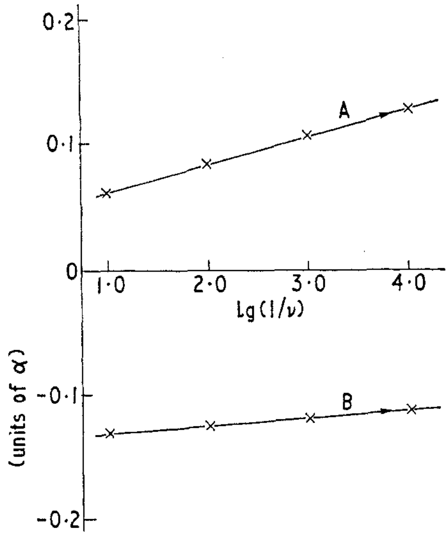

Figure 7. $\delta\rho/\rho$ (curve A) and $\delta\lambda/\lambda$ (curve B) against lg $(1/\nu)$.

In the P-wave bound-state the cut-off parameter Λ is $\Lambda = 1\mu_\pi$.

We now observe in figure 6 that for the 'heavy-photon' exchange, the mass-shift $\delta\rho/\rho$ is still dominated by the driving term but takes the opposite sign. Writing

$$\delta\rho \simeq \frac{\langle \delta V_1{}^c \rangle}{(1 - \langle \partial V_1 / \partial m_e \rangle)}$$

where $\langle \partial V_1/\partial m_e \rangle = 1{\cdot}16$ we have a 'sign reversal' in the mass shift coming from the effect of the perturbation on the strong interaction dynamics.

In view of the fact that the feedback coefficient $\langle \partial V_1/\partial m_e \rangle$ is a function of the cut-off Λ, it is an obvious question to ask whether the sign-reversal mechanism vanishes for some particular range of Λ. Surprisingly, calculation shows that $\langle \partial V_1/\partial m_e \rangle$ is almost completely insensitive to Λ.

Figure 7 shows the 'infrared' divergence for $\nu \to 0$ as in the S-wave case.

9. Conclusions

We have demonstrated that, in the perturbation of bootstrapped scalar and vector bound-state mesons generated from a relativistic Schrödinger equation, the sign of the eventual change in the binding energy, with the same perturbing potential, depends crucially on the dynamical structure of the system.

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Appendix 1. Computing remarks

The mapping of all the $l = 0$ integrals is

$$k = \mu_\pi \left(\frac{1+y}{1-y}\right), \qquad k\phi_0(k) = X(y).$$

The integral equation (4) then becomes

$$\lambda_{\mathrm{in}}^{-1} X(y) = \int_{-1}^{1} k(y, y')\, X(y')\, \mathrm{d}y'.$$

Use of Gaussian n-point quadratures leads to

$$\det(A - \lambda_{\mathrm{in}}^{-1} I) = 0$$

with A an $n \times n$ matrix $A_{ij} = \mathscr{R}_j k(y_i, y_j)$. $\mathscr{R}_j$ and y_i are chosen according to Gauss's method. We then write

$$\det(A - \lambda_{\mathrm{in}}^{-1} I) = (\lambda_{\mathrm{in}}^{-1})^n - \sum_{j=1}^{n} P_j (\lambda_{\mathrm{in}}^{-1})^{n-j} = 0$$

where the P_j are calculated by the method of Leverrier–Faddeev (Fadeev and Fadeeva 1963). For the P-wave integrals the mapping is

$$k = \tfrac{1}{2}\Lambda(1+y), \qquad k\phi_1(k) = X(y)$$

the rest of the numerics being the same as above for the S-wave case.

Appendix 2

We prove that, if the operator $(\hat{U}_l + \hat{W}_l)$ of equation (12) is singular and Hermitian, then the equation has solutions only if $\langle \phi_l | Q_l \rangle = 0$.

Proof

Put

$$(\hat{U}_l + \hat{W}_l) = \hat{M}_l (= \hat{M}_l^\dagger)$$

where † denotes Hermitian. Then we have

$$\begin{aligned} \langle \phi_l | Q_l \rangle &= \langle \phi_l | \{ \hat{M}_l | \delta\phi_l \rangle \} \\ &= \{ \langle \phi_l | \hat{M}_l \} | \delta\phi_l \rangle \\ &= 0 \quad \text{since} \quad \hat{M} | \phi_l \rangle = 0 \end{aligned}$$

which was to be demonstrated. Detailed inspection of the Q_l function shows that for all 'photon' masses we do indeed have $\langle \phi_l | Q_l \rangle \equiv 0$. Hence there is always a solution to equation (12) but it is not unique. We therefore calculate two independent solutions $u_1{}^l$ and $u_2{}^l$ and write $\delta\phi_l$ as a linear combination, namely

$$\delta\phi_l = \beta_l u_1{}^l + (1 - \beta_l) u_2{}^l.$$

Imposing the condition that $\langle \phi_l | \delta\phi_l \rangle = 0$ we fix β_l uniquely.

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